An introduction to knot theory

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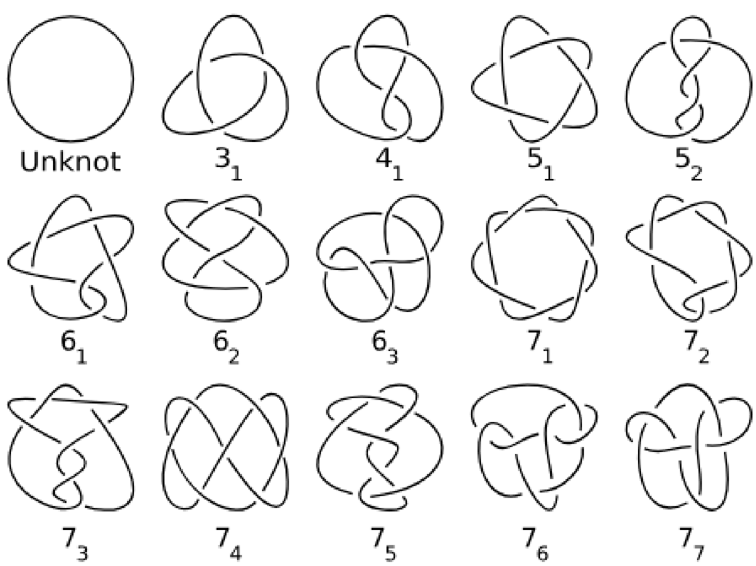
Sources and images are linked at the end of the article.

What is knot theory?

Knot theory is a branch of topology concerned with studying mathematical knots. A knot you may be familiar with is the one you tie on your shoelaces; however, mathematical knots differ as, unlike common knots, they must form a closed loop and have a continuous structure. For centuries, mathematicians have been studying knots, with the central problem being the knot equivalence problem. Through continued refinement of this problem, many different methods of distinguishing two knots from one another have been developed, and in this article, I will discuss some of the techniques. Additionally, I will explore how knot theory has evolved from a branch of mathematics that could have easily been mistaken for knowledge for its own sake into a framework utilised in many fields to develop a greater understanding of many areas of science.

An image to show examples of mathematical knots:



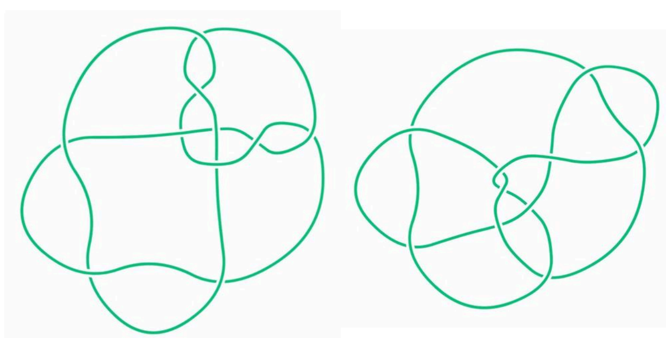
An image showing the first seven prime knots, with up to 7 crossings.
Note: Mathematical knots are commonly represented in either 2D or 3D

The knot equivalence problem

From the first tabulation of prime knots by mathematician Peter Guthrie Tait, a central problem was distinguishing knots. Initially, tabulation was done by hand, and prime knots were painstakingly discovered, notably by mathematicians Thomas Kirkman and Charles Little, who continued the work on Tait.

In Tait's paper “On Knots,” the first mathematical publication to mention knots, he said, “I cannot be *absolutely* certain that all those groups are essentially different from one another.” This was the first uncertainty shown and the moment that drove the development of knot equivalence, and numerous methods of distinguishing knots have been developed since then. I will discuss some in the next section.

The most well-known example that exemplified the struggle of distinguishing knots was the Perko pair. Initially, two knots in Tait’s tabulation were thought to be different from one another and so were listed as separate knots. However, in the late 20th century, Kenneth Perko discovered that these two knots were, in fact, identical but appeared the same under their projections. This highlighted the complexities and difficulties associated with distinguishing knots.

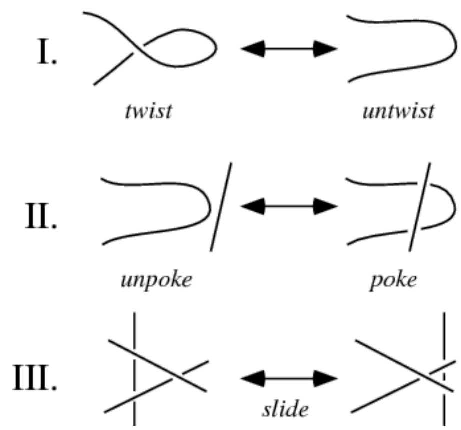


An image showing the Perko Pair, projected in 2D.

Reidermeister moves

One of the first advancements to the knot equivalence problem, Reidemeister Moves, were pioneered by Kurt Reidermeister in the early 20th century. By performing a series of moves, which could be a twist, slide, or poke, you can manipulate the shape of a knot without changing it. This allowed for the precise comparison of two knots, and if you could use a series of Reidemeister Moves to transform one knot into another, it confirmed that the two knots were equivalent.

Below are diagrams illustrating Reidemeister Moves and how they could be applied.



Note: For Reidemeister Moves to be applied, the knot must form a closed loop. The above diagram does not feature that, but it is purely for illustrating how the moves work and can be applied.

To perform Reidemeister Moves, a 3D knot has to be represented in a 2D diagram. As in the images above, this allows for a more precise representation of the 2 knots you are comparing. Furthermore, when applying Reidermeister moves to compare 2 knots, you cannot cut and reconnect a knot to change the knot.

While this development progressed the knot equivalence problem, it was slow and laborious to perform. Mistakes are easy to make when comparing 2 knots using Reidemeister moves, complicating the comparison.

Knot invariants

A notable progression to solving the knot equivalence problem was the development of knot invariants. A knot invariant is a property of a knot that remains unchanged, no matter how the knot may be changed or moved around. These have helped progress the knot equivalence problem, as they provided a method to classify knots by their invariants. I will discuss some of the most well-known invariants below.

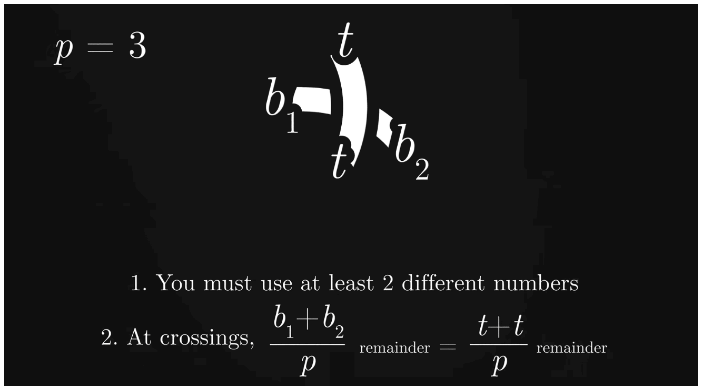
P-Colourability:

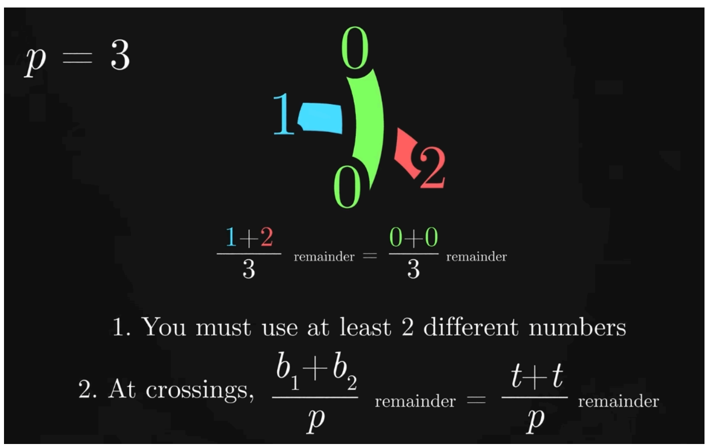
P-Colourability is a simple yet powerful knot invariant. For this invariant to work, the value of P can be any prime number besides 2. When applying p-colourability to a knot, you number each strand of the knot with numbers between 0 and P-1. You must also be sure to follow two rules:

1) You must use at least two different numbers to number your strands.
2) At crossings between strands, the sum of the two bottom strands divided by P must give the same remainder as twice the value of the top strand divided by P.

If these conditions are met, the knot is said to be P-colourable, and you can use the rules to assign numbers to each strand of a knot.

Here is an example of P-colourability on the trefoil knot. The trefoil is 3-colourable, meaning that the knot strands can be labelled 0,1 and 2 when applying the above three rules. However, an unknot cannot be three-coloured, consequently distinguishing the 2 knots.





P-Colourability provides a systematic method of analysing and distinguishing knots, which has helped mathematicians classify and tell them apart.

Alexander Polynomial:

Whilst P-colourability has provided a strong base for mathematicians to distinguish knots, some of the more powerful invariants have been polynomials; the Alexander Polynomial is an example.

The polynomial associates each knot "K" with a polynomial in a variable "t", and equivalent knots will have the same Alexander polynomial. This can be used as a tool and applied to any knot you may like, for example:

Note: Δ stands for the Alexander Polynomial, and the subscript indicates for which knot the polynomial has been calculated.

$$\Delta_{unknot} = 1$$

$$\Delta_{trefoil} = t-1+t^{-1}$$

The fact that the Alexander polynomial of the unknot differs from the Alexander polynomial of the trefoil tells us that the two knots are different.

However, this polynomial, in particular, has limitations, too. While more potent than P-colorability, the Alexander polynomial cannot always distinguish between knots. For instance, two distinct knots can have the same Alexander polynomial, such as the square and granny knots, which are not equivalent but share the same polynomial. This limitation has been resolved over many decades by introducing newer and more advanced polynomials, such as the Jones polynomial.

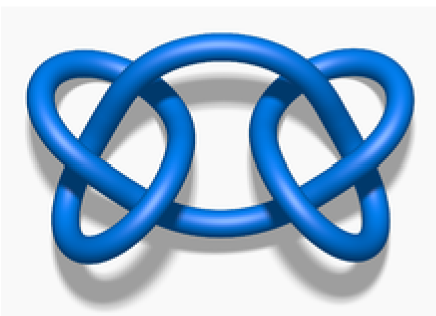


Image of the Square Knot
$\Delta_{squareknot} = (t-1+t^{-1})^2$

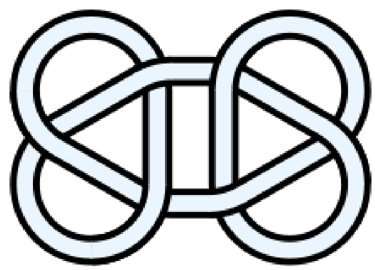


Image of the Granny Knot
$\Delta_{grannyknott} = (t-1+t^{-1})^2$

Above are images and descriptions showing how the granny and square knot have identical Alexander polynomials. This exemplifies the limitation of using only one invariant at a time, as two knots which are different, may be classified as identical with only one invariant.

<u>Summary:</u>

Knots have numerous invariants, two of which have been discussed above. Mathematicians compile these invariants and apply them to knots. This provides each knot with various differing sets of invariants that can be used to identify one knot from another.

Knot theory today

Today, Knot Theory is a more mature branch of mathematics. Mathematicians such as John Conway tabulated all 11 crossing knots by hand, and since then, computer algorithms have been created to tabulate all knots with 12,13 and up to 19 crossing knots. Currently, there are 352,152,252 prime knots known to us.

However, as the knot theory progressed, its applications became more tied to the world around us than simply pure mathematics.

Applications of knot theory

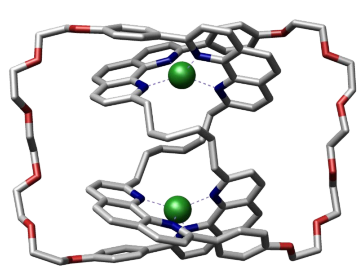
Recently, Knot Theory has been applied to numerous fields and has been found helpful in making discoveries and pioneering understanding of scientific concepts, some of which are discussed below.

DNA:

Knot theory has significant applications in the study of DNA. DNA strands often become tangled and knotted during cellular replication and transcription. Knot theory provides a framework for understanding and classifying these entanglements, which is critical for analysing how enzymes, like topoisomerases, resolve these knots by cutting and rejoining the strands. Understanding the knotting behaviour of DNA is vital for studying genetic stability and preventing errors that can lead to diseases such as cancer. Knot theory has helped to understand how this process works and provided a platform to build upon in developing more treatments or knowledge in this field.

Molecular Knots:

In 1989, chemist Jean-Pierre Sauvage tied molecules around copper ions to create the first-ever synthetic knotted molecule. This knotted molecule was based on a trefoil, which restricted the atoms from moving or changing orientation, giving the molecule new properties. This was a particularly exciting discovery, as it opened the door to using knot theory to create new molecules that could have beneficial properties in different applications.



Above is an image of the knot which Sauvage created.

After the trefoil, scientists could only tie five other molecular knots due to the difficulty of the task. This is because molecules must be built to self-assemble into knots. However, it is evident that molecular knots have potential in fields such as materials science and nanotechnology, where the properties that the knots may give the molecules, such as mechanical strength are crucial.

Conclusion

Knot theory has transitioned from a niche area of mathematics with limited applicability to a framework that improves understanding across various disciplines. As the field has matured and the knowledge surrounding it has been refined, it has established itself as a tool for advancing knowledge in numerous domains. Given this trajectory, knot theory will continue to be key in future discoveries, contributing to mathematical research and its applications in science and technology.

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