Bridges between Mathematics and the World of Finance

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Introduction:

1 There are a multitude of fields in mathematics that have many practical uses within the financial industry, ranging from algebra and calculus to numerical methods and statistics. The two most well-known applications involve actuarial science and quantitative finance; actuaries mainly utilise statistics and probability to evaluate and manage risks in finance, e.g., within the insurance sector, while the latter revolves heavily around the development of advanced mathematical models to predict and comprehend the behaviour of financial markets. Quantitative analysts apply many of these models within financial institutions to price options, manage potential risks, and enhance portfolios. An example of one of these models (describing the fair price of options) is the Heston model, which is a variant of the famous Black-Scholes model that will be discussed below.

Heston Model:

2 The Black-Scholes model is fundamentally a partial differential equation with 6 input variables: volatility, time to expiration, risk-free interest rate, underlying stock price, strike/exercise price, type of option. These variables are denoted by **σ, t, r, St, X,** and **C** *&* ***P*** respectively, with **C** representing the call option and **P** the put option. Sigma is used to denote volatility as it is measured as a standard deviation of logarithmic returns, which in simple terms is the variance of a trading price over time. This element is the key difference that the Heston variation evolves from as in the original Black-Scholes model volatility is kept constant, whilst in the variation it can fluctuate. This leads to the stochastic differential equations below from which PDEs can be derived:

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dW_1(t)$$

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dW2(t)$$

Note: Here $\sqrt{v(t)}$ now stands for volatility as σ now stands for the volatility of the volatility

The first SDE explains the price movement of the asset (how the expected return and fluctuations are influenced by the volatility), while the second explains the fluctuations of the actual volatility itself as a process. There are three new variables introduced with theta(**θ**) denoting the long-term average volatility, kappa(**κ**) the mean reversion rate (back to θ) and **W** the wiener process. This connects another bridge into the interdisciplinary connection stemming from physics. The Wiener process is a mathematical model for Brownian motion and is a stochastic process that demonstrates the volatility of asset prices as they can randomly fluctuate and not take any negative values similar to the movement of particles in a fluid.

Computation:

3 PDEs from the Heston model can be computed either semi-analytically or computationally using numerical methods; however, another parameter is essential in the form of ρ (the correlation coefficient between the asset and its variance (***d*****S(t) &** ***d*****v(t)**)). An example of a numerical method that can be used is finding the characteristic function of the Heston model, subsequently using the inverse Fourier transform to produce an integral that can then be solved by implementing the trapezoidal rule using a step size h, which provides a closed-form solution. Another involves the use of the Laplace Transform which is another example of a mathematical technique of which can be so useful in the financial world allowing the complex PDE to be transformed with respect to time into a much more simple ODE which could then give way to a solution. Through generalised finite integration the system could be solved and would then have to be converted back to the original time domain using an inverse Laplace transform. On the other hand, a computational approach may be applied through the use of Python by taking advantage of various packages such as NumPy for the numerical computations, SciPy for numerical integration, and Matplotlib for plotting and visualising the results from solving these PDEs.

Arbitrage:

4 There are three key concerns within finance: don't be exploited, understand and manage risk, and generate profit. With regards to exploitation, an arbitrage is the most basic investment that traders capitalise on in order to create a source of risk-free profit. An arbitrage refers to the opportunity to make this type of profit through buying and selling assets in different markets or forms to exploit price differences. It is considered risk-free as it costs nothing to set up at time t (i.e., $Y_t \leq 0$) but in the future, at time T, the asset has no probability of being negative (i.e., $P(Y_t<0)$) so the payoff can only be positive. The Heston model utilises a no arbitrage condition, making sure any option prices derived don't have any arbitrage opportunities as this would destabilise the market. Mathematics ensures that any financial models produced generate realistic, consistent prices by embedding this principle and this avoids any potential market discrepancies. In summary, mathematical models are designed with assumptions that allow them to appropriately reflect market dynamics, allowing reliable and practical closed-form solutions, effective risk management strategies, and overall market consistency.

Constrained Optimisation:

5 Mathematics can also be applied to optimise a portfolio's expected return while meeting certain risk thresholds and budget constraints (typically within the risk and asset management sector). A frequently used analytical technique for constrained optimisation is the Lagrange multiplier (λ) which derives from two functions. The first one is what you want to optimise with respect to multiple variables (f(x,y,...)), which is subject to the constraints of the second function equalling 0 (g(x,y,,,,)=0). Using the Lagrangian Multiplier, the two functions can be combined into one, which is the Lagrange function, and its gradient (partial derivatives) can be set to equal zero. Thus allowing a system of equations to be formed that can be algebraically solved to find the optimal values of the variables, multiplier and solution.

A very simple example would be as follows:

A skincare brand wants to optimise the revenue in the production of their sunscreen. Letting **h** be the number of hours of labour and **p** the number of products manufactured. Labour costs £780/hour, and the average material price is £4/product. The budget per day is £15000 where workers only operate 12 hours a day. The revenue can be modelled by the multivariable function $f(h,p)=100h^2p$.

$780\mathbf{h}+4\mathbf{p}= 15000 \Rightarrow g(\mathbf{h},\mathbf{p})= 780\mathbf{h}+4\mathbf{p}\ \mathbf{c}= 15000$

$\mathbf{L}(\mathbf{h},\mathbf{p},\lambda)= f(\mathbf{h},\mathbf{p})-\lambda(g(\mathbf{h},\mathbf{p})-15000)= 100\mathbf{h}^2\mathbf{p}-\lambda(780\mathbf{h}+4\mathbf{p}-15000)$

$\nabla\mathbf{L}= 0 \therefore \partial\mathbf{L}/\partial\mathbf{h} = \partial f/\partial\mathbf{h} - \lambda(\partial g/\partial\mathbf{h}) = 0 \Rightarrow 200\mathbf{hp} - \lambda(780) = 0$

$\partial\mathbf{L}/\partial\mathbf{p} = \partial f/\partial\mathbf{p} - \lambda(\partial g/\partial\mathbf{p}) = 0 \Rightarrow 100\mathbf{h}^2 - \lambda(4) = 0$

$\partial\mathbf{L}/\partial\lambda = -(g(\mathbf{h},\mathbf{p})-15000) = 0 \Rightarrow 780\mathbf{h}+4\mathbf{p} = 15000$

Solving these simultaneously would result in an optimal solution of **≈12.82** hours per day worth of labour with approximately **1250** products manufactured and a Lagrangian multiplier of 4109.14. The maximum revenue per day would evaluate to approximately **£20,544,000** for the company (a realistic function is not used). Using these results, the mathematician would suggest increasing their production times by one hour from 12 to 13 hours so that maximal profit can be achieved as 12<12.82<13. More complex problems can be fed into computers using the Lagrangian function to optimise functions such as revenue with numerous variables in a real-world application. The Lagrangian multiplier itself is often referred to as a shadow price of a certain constraint, meaning that it represents how much the return would improve or decrease if the constraints were slightly loosened.

Linear algebra in machine learning, fluid dynamics in predicting weather patterns, and number theory in designing encryption algorithms—there are countless other applications of advanced mathematical concepts in finance and many other industries that can give way to new technological advancements.

References:

[1] The Economic Times. (Last Updated: 2024) *What is Black-Scholes Model? definition of black-scholes model, Black-Scholes model meaning*. [online]. Available From: https://economictimes.indiatimes.com/definition/black-scholes-model. [Accessed: 11/08/2024].
[2] Khan Academy. (2013) *Introduction to the Black-Scholes formula | Finance & Capital Markets | Khan Academy*. [online]. Available From: https://www.youtube.com/watch?v=pr-u4LCFYEY. [Accessed: 12/08/2024].
[3] Neisy, Peymany. (2011) *Financial Modeling by Ordinary and Stochastic Differential Equations.* [online]. Available From: https://www.idosi.org/wasj/wasj13(11)/5.pdf. [Accessed: 12/08/24].
[4] Rouah, F.D. (?) *Simplified Derivation of the Heston Model 1 Heston Dynamics 2 The Heston PDE.* [online]. Available From: https://www.frouah.com/finance%20notes/The%20Heston%20model%20short%20version.pdf. [Accessed: 13/08/24].
[5] Engineering Analysis with Boundary Elements. (2018) *Generalised finite integration method for solving multi-dimensional partial differential equations.* [online]. Available From: https://www.sciencedirect.com/science/article/abs/pii/S0955799718304284. [Accessed: 14/08/24].
[6] Springer Finance. (2nd Ed: 2011) *The Mathematics of Arbitrage.* [online]. Available From: https://beckassets.blob.core.windows.net/product/readingsample/264513/9783540219927_excerpt_001.pdf. [Accessed: 15/08/24].
[7] Cohen, S. (2020) *Oxford Mathematics 3rd Year Student Lecture - Mathematical Models of Financial Derivatives.* [online]. Available From: https://podcasts.ox.ac.uk/oxford-mathematics-3rd-year-student-lecture-mathematical-models-financial-derivatives. [Accessed: 15/08/24]
[8] Oxford Reference. (?) *Lagrange multiplier.* [online]. Available From: https://www.oxfordreference.com/display/10.1093/oi/authority.20110803100047903. [Accessed: 16/08/24]